# Fourier Analysis Mar 3, 2022

Review.

Thm (Weyl).

Let $\gamma$ be an irrational number. Then the sequence

$$\left(\{n\gamma\}\right)_{n=1}^{\infty}$$

is equidistributed in $[0,1)$.

Recall that a sequence $(x_n)_{n=1}^{\infty} \subset [0,1)$ is said to be equidistributed in $[0,1)$ if for any $(a,b) \subset [0,1)$,

$$① \quad \lim_{N\to\infty} \frac{1}{N} \#\left\{ 1 \leq n \leq N : \quad x_n \in (a,b) \right\} = b-a.$$

Equivalently, ① can be rewritten as

$$② \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(x_n) = b-a = \int_0^1 \chi_{(a,b)}(x)\,dx$$

where $\chi_{(a,b)}(x) = \begin{cases} 1 & \text{if} \quad x \in (a,b) \\ 0 & \text{if} \quad x \in [0,1) \setminus (a,b) \end{cases}$

Extend $\chi_{(a,b)}$ to be a 1-periodic function on $\mathbb{R}$.

Then Weyl Thm says that

$$③ \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) = \int_0^1 \chi_{(a,b)}(x)\,dx.$$

In what follows we prove ③.

Lem 2. Let $f$ be a 1-periodic continuous function on $\mathbb{R}$. Then

$$④ \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\gamma) = \int_0^1 f(x)\,dx,$$

where $\gamma$ is an irrational number.

Pf. We divide the proof into 3 steps.

Step 1. We prove ④ if $f$ is of the form

$$f(x) = e^{2\pi i k x}, \quad k \in \mathbb{Z}.$$

When $k=0$, then $f \equiv 1$. Then ④ clearly holds.

When $k \neq 0$,

$$\frac{1}{N} \sum_{n=1}^{N} f(n\gamma) = \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k n \gamma}$$

$$= \frac{1}{N} \sum_{n=1}^{N} \left(e^{2\pi i k \gamma}\right)^n$$

$$= \frac{1}{N} e^{2\pi i k \gamma} \frac{\left(1 - e^{2\pi i k N \gamma}\right)}{1 - e^{2\pi i k \gamma}}$$

Hence

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(n\gamma) \right| \leq \frac{2}{N \left| 1 - e^{2\pi i k \gamma} \right|} \to 0 \text{ as } N \to \infty$$

(Notice that $e^{2\pi i k \gamma} \neq 1$ since $\gamma$ is irrational)

Meanwhile $\int_0^1 f(x)\,dx = 0$

So ④ holds in this case.

Step 2. Notice that if $f, g$ satisfy ④, then so is $\alpha f + \beta g$ for all $\alpha, \beta \in \mathbb{C}$.

As a consequence, ④ holds if $f$ is

a trigonometric polynomial of the form

$$\sum_{n=-N}^{N} c_n e^{2\pi i n x}.$$

Step 3. ④ holds for all 1-periodic cts functions on $\mathbb{R}$.

Let $f$ be a 1-periodic cts function on $\mathbb{R}$.

Then $\forall \varepsilon > 0$, by the Weierstrass approximation Thm,

$\exists$ a trigonometric polynomial

$$g(x) = \sum_{n=-N}^{N} c_n e^{2\pi i n x}$$

such that

$$|f(x) - g(x)| < \varepsilon \quad \text{for all } x \in \mathbb{R}.$$

Now

$$\left| \frac{1}{N} \sum_{n=-N}^{N} f(n\gamma) - \int_0^1 f(x)\,dx \right|$$

$$\leq \frac{1}{N} \sum_{n=-N}^{N} |f(n\gamma) - g(n\gamma)| + \int_0^1 |f(x) - g(x)|\,dx$$

$$+ \left| \frac{1}{N} \sum_{n=-N}^{N} g(n\gamma) - \int_0^1 g(x)\,dx \right|$$

$$\leqslant 2\varepsilon + \left| \frac{1}{N} \sum_{n=-N}^{N} g(n\gamma) - \int_0^1 g(x)\,dx \right|$$

$$\leqslant 3\varepsilon \quad \text{if } N \text{ is large enough.}$$

This proves ④. $\boxtimes$

Pf of Weyl's Thm; i.e. ③ holds.

Finally, we prove ③.

i.e. $$\frac{1}{N} \sum_{n=-N}^{N} \chi_{(a,b)}(n\gamma) \to \int_0^1 \chi_{(a,b)}(x)\,dx.$$

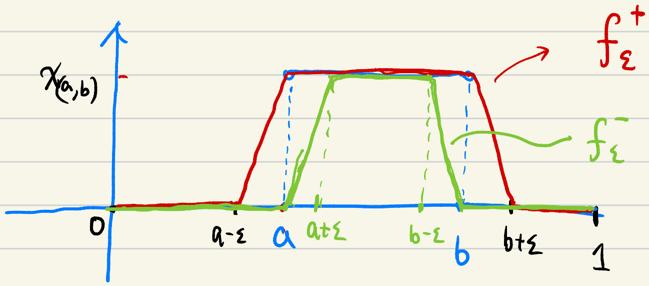
To prove this, for any $\varepsilon > 0$, we construct two 1-periodic cts function $f_\varepsilon^+$, $f_\varepsilon^-$ such that

$$f_\varepsilon^- \leqslant \chi_{(a,b)}(x) \leqslant f_\varepsilon^+$$

and

$$\int_0^1 \left( f_\varepsilon^+(x) - \chi_{a,b}(x) \right) dx < 2\varepsilon$$

$$\int_0^1 \left( \chi_{(a,b)}(x) - f_\varepsilon^-(x) \right) dx < 2\varepsilon.$$




Then

$$\overline{\lim_{N\to\infty}} \frac{1}{N}\sum_{n=-N}^{N} \chi_{(a,b)}(n\gamma) \leq \overline{\lim_{N\to\infty}} \frac{1}{N}\sum_{n=-N}^{N} f_\varepsilon^+(n\gamma)$$

$$= \int_0^1 f_\varepsilon^+(x)\,dx$$

$$= \int_0^1 \chi_{(a,b)}(x)\,dx + 2\varepsilon$$

Letting $\varepsilon \to 0$ gives

$$\underline{\overline{\lim_{N\to\infty}} \frac{1}{N}\sum_{n=-N}^{N} \chi_{(a,b)}(n\gamma)} \leq \int_0^1 \chi_{(a,b)}(x)\,dx$$

Similarly,

$$\underline{\lim}_{N\to\infty} \frac{1}{N}\sum_{n=-N}^{N} \chi_{(a,b)}(n\gamma) \geq \underline{\lim}_{N\to\infty} \frac{1}{N}\sum_{n=-N}^{N} f_\varepsilon^-(n\gamma)$$

$$= \int_0^1 f_\varepsilon^-(x)\,dx \geq \int_0^1 \chi_{(a,b)}\,dx - 2\varepsilon$$

Hence

$$\underline{\lim}_{N\to\infty} \frac{1}{N} \sum_{n=-N}^{N} \chi_{(a,b)}(n\gamma) \geq \int_0^1 \chi_{(a,b)}(x)\, dx .$$

$\square$